CS 715: Advanced Topics in Algorithmic Game Theory and Mechanism Design

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Lecture 5

Agenda

- Optimal Mechanism Design
  - Matroids
  - Framework
  - Sequential pricing for single parameter (BSMD)
  - Order Oblivious Posted Pricing
  - Multi-parameter setting (BMUMD)
  - Reduction from BMUMD to BMSD

Matroid

Let E be a finite set, and $\mathcal{I}$ be the collection of subsets of E

Definition

$\mathcal{M} = (E, \mathcal{I})$ is called a matroid if

1. $\phi \in \mathcal{I}$
2. if $A \in \mathcal{I}$, $A' \subset A \in \mathcal{I}$
3. if A and $B \in \mathcal{I}$ and $|A|$ ¡ $|B|$, then, $\exists x \in B$ s.t. $A \cup \{x\} \in \mathcal{I}$.

For Example:
uniform matroid
forest matroid

Single Parameter Framework

- $J = \{1, 2, \ldots, m\}$: Set of Services. i^{th} service is targeted for the agent i. $(m = n)$
- S: feasibility constraints, $S \subset 2^J$
- F: Distribution of agent's valuation
- Designing mechanism for (J, S, F) is Bayesian Single-parameter Mechanism Design Problem (BSMD)

For example: single item auction

Posted Price Mechanism

Given prices $\mathbf{p} = (p_1, \ldots, p_n)$ and order σ over services, posted price mechanism is defined as:

Definition

1. $A = \phi$
2. $j = 1 : m$
   1. If $A \cup \sigma(j) \in S$, offer service $\sigma(j)$ at price p_j
   2. If agents accepts, $A = A \cup \sigma(j)$.

Revenue in Sequential Price Mechanism (SPM)
$R^{(\mathbf{p},\sigma)}_{(J,S,F)} = \mathbf{E}_{v \sim F}[R^{(\mathbf{p},\sigma)}_{(J,S,F)}(v)]$
Revenue in Order Oblivious Posted Price Mechanism (OPM)
$R^{(\mathbf{p},\sigma)}_{(J,S,F)} = \mathbf{E}_{v \sim F}[\min_\sigma R^{(\mathbf{p},\sigma)}_{(J,S,F)}(v)]$

Multi-parameter Mechanism Design

- $J = \{1, 2, \ldots, m\}$: Set of Services.
- $[n] = \{1, 2, \ldots, n\}$: Set of agents.
- $\Pi = \{J_1, \ldots, J_n\}$, partition of J. The services in J_i are targeted for i.
- S: feasibility constraints, $S \subset 2^J$ and $|S \cap J_i| \leq 1$ (Unit Demand).
- F: Distribution of agent's valuation
- Designing mechanism for (J, S, Π, F) is Bayesian Multi-parameter Unit demand, Mechanism Design Problem (BMUMD)
- BSMD is BMUMD with $J_i = \{i\}$.

Sincere Strategy

- Sincere strategy: for an agent when offered a service j s.t $p_j \leq v_j$, then accept the offer, else reject.
- Desirable: Sincere strategy is dominant strategy.
- Ordering σ is J_i respecting if, $j_1, j_2 \in J_i$ and $v_{j_1} - p_{j_1} > v_{j_2} - p_{j_2} \geq 0$, then $\sigma^{-1}(j_1) < \sigma^{-1}(j_2)$.
- σ is Π respecting if it is J_i respecting for all $i \in [n]$

Lemma

Sincere is dominant strategy in a posted price mechanism $(\mathbf{p}, \sigma)$ iff σ is Π respecting.

Proof?

Revenue in OPM:

$$R^{(\mathbf{p},\sigma)}_{(J,S,\Pi,F)} = \mathbf{E}_{v\sim F}[\min_{\sigma:\sigma \text{ is } \Pi \text{ respecting}} R^{(\mathbf{p},\sigma)}_{(J,S,\Pi,F)}(v)]$$

Partial Dominant Strategy Implementation

- How to ensure σ is Π respecting?
- Need not be possible always
- Hence some agents may have sincere as dominant strategy and other agents need not.

Definition

A mechanism is a Partial Dominant Strategy implementation of a desired objective if the objective is met whenever every agent with weakly dominant strategy plays that strategy irrespective of other agents' strategies.

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For BSMD,

Lemma

If F_i is regular, the revenue of any incentive compatible mechanism M is bounded above by $\sum_i p_i^M q_i^M$ where q_i^M is the probability with which agent i is served and $p_i = F_i^{-1}(1 - q_i^M)$.

Proof.

BMUMD to BSMD

$\mathcal{I} = (J, S, \Pi, F) \Rightarrow \mathcal{I}^{rep} = (J, S, F)$ each service is distinct agent. (Example in the class).

Lemma

For any IR and IC A, $R^A_{\mathcal{I}} \leq R^{Mye}_{\mathcal{I}^{rep}}$.

Theorem

If OPM $\mathbf{p}$ *is* α*-approximate to an optimal mechanism for* $\mathcal{I}^{rep}$*, then it is* α*-approximate for* $\mathcal{I}$*. That is, if* $R^P_{\mathcal{I}^{rep}} \geq \frac{1}{\alpha} R^{OPT}_{\mathcal{I}^{rep}}$ *then* $R^P_{\mathcal{I}} \geq \frac{1}{\alpha} R^{OPT}_{\mathcal{I}}$

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Partition Matroid

Lemma

Let $\mathcal{I}$ be an instance of an BSMD with S being intersection of two partition matroids, then there exists a set of prices $\mathbf{p}$ s.t. $R^P_{\mathcal{I}} \geq \frac{4}{27} R^{OPT}_{\mathcal{I}}$

- For k-uniform matroids, there exists 2-approximation pricing
- For arbitrary constraints, any OPM has approximation $\Omega(\frac{\log n}{\log \log n})$.

Note that the approximation guarantee of reduction does not hold true for SPMs.

Summary

Table: OPM

S (Feasibility Constraints)	Upper Bound	Lower Bound
General Matroid	$O(\log k)$	2
Uniform Matroid, Partition Matroid	*2	2
Graphical Matroid	*3	2
Intersection of two Partition Matroids	*6.75	2
Non Matroid	-	$*\Omega\frac{\log n}{\log\log n}$

Table: SPM

S (Feasibility Constraints)	Upper Bound	Lower Bound
General Matroid	*2	$\sqrt{\pi/2} \approx 1.25$
Uniform Matroid, Partition Matroid	$e/e-1 \approx 1.58$	1.25
Intersection of two Matroids (BSMD)	3	1.25
Intersection of two Matroids (BMUMD)	*8	1.25
Non Matroid	-	$*\Omega\frac{\log n}{\log\log n}$

* In STOC'10 paper